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The effect of the overtaking disturbance on a shock wave moving in a non-uniform medium

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When a shock moves in a non-uniform medium, its motion is influenced both by the non-uniformity ahead of the shock and also by a wave which overtakes the shock. In this paper the overtaking wave is examined in an analytic manner by considering its formation and subsequent propagation. This analysis is then combined with the known 'freely propagating' description of the shock motion, in which the overtaking wave is ignored, and results in a description of the shock motion in integral form. For the particular problem of a strong shock propagating in a medium with a power-law density variation, the freely propagating shock law has previously been compared with the available similarity solution. The estimate of the effect of the overtaking wave presented in this paper is shown to provide a significant improvement to the description of the shock motion in this instance.

1. Introduction

This paper is concerned with the one-dimensional motion of a strong shock wave moving into a medium which is at rest and at uniform pressure, but has a variable density. One of the authors (Chisnell 1955) determined a simple law of propagation for the shock by taking account of the interaction of the shock with the density variation ahead of the shock, but ignoring the effect on the shock of disturbances from the non-uniform flow behind the shock. This simple description of the motion of the shock will be referred to as 'freely propagating'. The same result was obtained in a more elegant manner by Whitham (1958), by applying the differential relation valid along an overtaking characteristic to the flow variables behind the shock.

Sakurai (1960) considered the motion of a shock moving outwards through a star. The shock is considered to be plane and the density of the star varies according to a positive power law in distance from the star boundary. The shock becomes strong as it approaches the boundary of the star, and the strong shock equations permit a similarity solution in this neighbourhood. Nine cases of the similarity solution were given by Sakurai, corresponding to gases with ratios of specific heats $\gamma = 1{\cdot}2$, $1{\cdot}4$ and $\frac{5}{3}$ and a power-law density distribution with exponents $\frac{1}{2}$, 1 and 2. The freely propagating shock law was compared with the similarity solution and found to be in reasonably good agreement; the agreement was worst in the case $\gamma = 1{\cdot}2$ for a density exponent of 2, where the shock law exponent was in error by 11 %.

Yousaf (1978) found the strength of the overtaking wave numerically from the similarity solution, and considered its effect on the freely propagating shock law. He showed that the resulting modification to the freely propagating law brings it into exact agreement with the similarity solution.

The present paper studies the overtaking wave in an analytic manner for an arbitrary initial density distribution. In §2 a derivation of the freely propagating shock law is obtained by considering the interaction of the shock with an elementary density discontinuity and integrating the resulting differential relation between shock strength and density. At this interaction a reflected disturbance is generated which propagates along a characteristic into the flow behind the shock, and a density discontinuity of changed strength propagates along a particle path behind the shock. The flow behind the shock is determined by the interaction of the two waves formed by the reflected disturbances and the density discontinuities. These two waves interact and generate a third wave, which propagates along the other set of characteristics and overtakes the shock. The analysis of the interaction of the two waves is made tractable by making the assumption used in the derivation of the freely propagating shock law; namely that the overtaking wave may be ignored. This description is used in §3 to estimate the strength of the overtaking wave and its effect on the motion of the shock. The result is in the form of an integral; the integrand contains a factor which depends on the geometry of the characteristics, and this in turn depends on the choice of initial density distribution.

In §4 the approximate theory of the two previous sections is applied to the density power-law problem solved by Sakurai. The freely propagating shock law exponent and the modification to it produced by the overtaking wave are determined. For each of the nine cases considered by Sakurai, these two approximate exponents lie on either side of the exact exponent, with the error in the modified exponent typically six times as small as the error in the freely propagating exponent. It is possible that the shock path in problems for which no similarity solution exists may be shown to lie between the freely propagating shock path and a modified path determined by an approximate description of the overtaking wave.

2. The freely propagating shock motion

A shock wave moves in the positive x-direction into a medium at rest that has a uniform pressure but a non-uniform density $\rho_0(x)$. The motion of the shock is affected both by variations in the density ahead of the shock and by an overtaking wave. The overtaking wave propagates along C_+ characteristics and is produced by disturbances in the non-uniform flow behind the shock. In this section a simple description of the flow is obtained by considering the shock motion to be influenced only by the density variations ahead of the shock. The overtaking C_+ disturbances are suppressed, and the resulting description of the shock motion and the flow behind the shock is called freely propagating.

The freely propagating description of the flow is available for shocks of any strength. The application in §4 to a power-law density distribution relates to strong shocks. Accordingly we derive the shock propagation law using the simpler form of the Rankine–Hugoniot equations valid for strong shocks. In terms of the shock speed U, the density ρ_0 ahead of the shock, and the ratio of specific heats γ, these equations give the pressure, density, fluid and sound speeds behind the shock as

$$p = \frac{2}{\gamma+1}\rho_0 U^2, \quad \rho = \frac{(\gamma+1)}{(\gamma-1)}\rho_0, \quad u = \frac{2}{\gamma+1}U, \quad a = \frac{(\gamma-1)}{(\gamma+1)}sU, \tag{2.1}$$

where

$$s = \left(\frac{2\gamma}{\gamma-1}\right)^{\frac{1}{2}}. \tag{2.2}$$

When the shock encounters an incremental change $\delta\rho_0$ in the density of the medium there is an associated shock speed increment δU, and the increments in the flow variables behind the shock are given in terms of these two increments by (2.1). In an exact analysis these increments in pressure and fluid velocity are supported both by a C_- disturbance generated by the interaction and by an overtaking C_+ disturbance. In the freely propagating description used in this section the C_+ disturbance is suppressed, and the pressure and velocity increments are therefore related by

$$\delta p = -\rho a\,\delta u, \tag{2.3}$$

a relation valid across a C_- disturbance. Combining (2.3) with (2.1) in differential form gives

$$\beta\frac{\delta\rho_0}{\rho_0}+\frac{\delta U}{U} = 0, \quad \beta = \frac{1}{2+s}, \tag{2.4}$$

or $U = \kappa\rho_0^{-\beta}$, with κ a constant.

We now consider the disturbances in the flow behind the shock wave in the freely propagating description. For the C_- disturbances generated by the shock, the pressure and velocity increments occurring in (2.3) may be expressed in terms of ρ_0 by use of (2.1) and (2.4) as

$$\delta p_- = \frac{2U^2}{\gamma+1}(1-2\beta)\,\delta\rho_0, \quad \delta u_- = -\frac{2}{\gamma+1}\beta U\frac{\delta\rho_0}{\rho_0}. \tag{2.5}$$

The minus suffix denotes a C_- disturbance. The density increment across the C_- disturbance is

$$\delta\rho_- = \frac{1}{a^2}\delta p_-, \tag{2.6}$$

and the total density increment $\delta\rho$ behind the shock is given in terms of $\delta\rho_0$ by (2.1). The difference between these two density increments is developed across a contact discontinuity, or C_0 characteristic, which moves with the fluid and has strength

$$\delta\rho_C = \delta\rho-\delta\rho_- = \alpha\,\delta\rho, \quad \alpha = 1-\frac{1}{\gamma}(1-2\beta). \tag{2.7}$$

The C_- and C_0 disturbances, whose strengths when generated by the shock are given by (2.5) and (2.7), interact in the flow behind the shock and produce the overtaking C_+ wave which is considered in §3. To find the strength of either the C_-, or the C_0, disturbance at a general point in the flow, its departure from its shock-generated value is found by considering the sequence of interactions it undergoes with disturbances of the other type. In this section we need to discuss only the interaction of C_- and C_0 disturbances, but as it will be necessary to include a C_+ disturbance in §3 we consider the general interaction involving all three disturbances. Let elements of

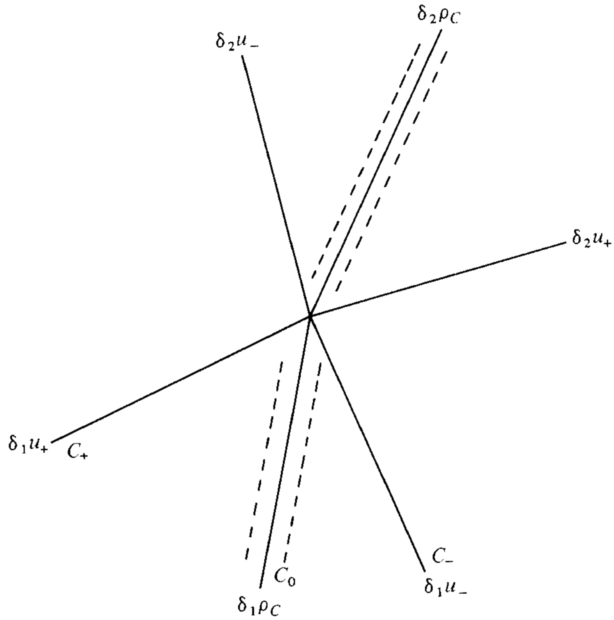

FIGURE 1. The general interaction at a point in the non-uniform flow behind the shock. The C_+, C_0, C_- disturbances have discontinuities $\delta_1 u_+$, $\delta_1 \rho_C$, $\delta_1 u_-$ across them before the interaction and $\delta_2 u_+$, $\delta_2 \rho_C$, $\delta_2 u_-$ after the interaction. In §2 the C_+ disturbance is suppressed. Also shown are two particle paths on either side of the C_0 disturbance; adjacent particles on these paths have the same velocity and are at the same pressure before they cross the C_+ and C_- disturbances and again after they have crossed the disturbances.

the C_-, C_0, C_+ waves have strengths $\delta_1 u_-$, $\delta_1 \rho_C$, $\delta_1 u_+$ before the interaction and $\delta_2 u_-$, $\delta_2 \rho_C$, $\delta_2 u_+$, respectively, after the interaction, as illustrated in figure 1. A fluid particle on one side of the C_0 characteristic experiences velocity increments $\delta_1 u_-$ and $\delta_2 u_+$ as it passes through the interaction region, whilst a particle on the other side of the C_0 characteristic experiences increments $\delta_1 u_+$ and $\delta_2 u_-$. The sums of these pairs of increments must be the same, as the velocity is continuous across a C_0 characteristic. The corresponding sums of pressure increments, given by (2.3) for a C_- disturbance and a similar equation without the minus sign for a C_+ disturbance, are also the same. These two relations determine the changes in strength of the C_- and C_+ disturbances as

$$\delta_2 u_- - \delta_1 u_- = \frac{1}{4\rho} \delta_1 \rho_C (\delta_1 u_- - \delta_1 u_+) = \delta_2 u_+ - \delta_1 u_+, \tag{2.8}$$

where ρ is the density between the incident C_- and C_0 disturbances. The density increments across the C_- and C_+ disturbances are given in terms of the pressure increments by (2.6), and a comparison of the density increments experienced by fluid particles on either side of the C_0 characteristic gives

$$\delta_2 \rho_C - \delta_1 \rho_C = \frac{1}{\rho} \delta_1 \rho_C (\delta_1 \rho_+ + \delta_1 \rho_-). \tag{2.9}$$

These interaction results may be used to determine the strengths of the C_- and C_0 disturbances at a general point P in the flow. In figure 2 the C_- disturbance leaves the shock at Q and the C_0 disturbance leaves at R. Equation (2.9) gives the change in

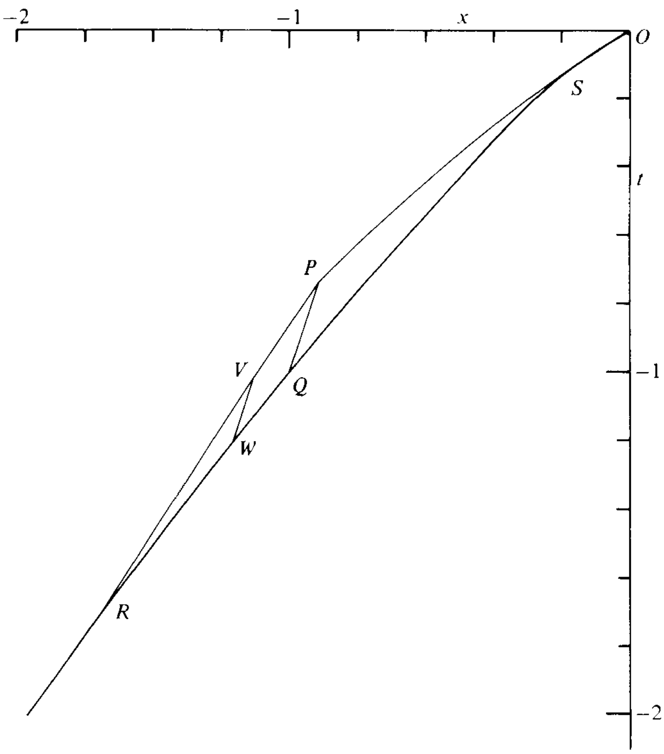

FIGURE 2. The shock path (R, W, Q, S, O) is shown for motion in a gas having $\gamma = 1{\cdot}4$ and a linear density gradient $\lambda = 1$. The arbitrary constants A, B of (4.2) have been chosen so that the shock passes through Q, with $x_Q = -1$, $t_Q = -1$. The C_- characteristic through Q and the C_+ characteristic through S, which has $t_S = -0{\cdot}1$, are shown meeting at P. The C_0 characteristic through P leaves the shock at R. The C_- characteristics QP and WV are straight lines, see (4.6).

the strength of the C_0 disturbance at any of its interactions between R and P. Along a particle path, $\delta_1\rho_+ + \delta_1\rho_-$, which occurs in (2.9), is the total density increment $\delta\rho$ encountered by the C_0 disturbance. Hence the equation may be recast as

$$\delta(\delta\rho_C) = \frac{1}{\rho}\,\delta\rho_C\,\delta\rho, \tag{2.10}$$

and gives the change in strength of the disturbance at a general point of its path. Integration from R to P gives the strength of the C_0 disturbance at P as

$$\left.\begin{aligned}\delta\rho_C(P) &= \frac{\rho(P)}{\rho(R)}\,\delta\rho_C(R)\\ &= \alpha\frac{\rho(P)}{\rho(R)}\,\delta\rho(R),\end{aligned}\right\} \tag{2.11}$$

by use of (2.7). This result does not depend on the freely propagating assumption, but it should be noted that the derivation remains valid if the assumption is used by putting $\delta_1\rho_+$ in (2.9) equal to zero.

To determine the strength of the C_- disturbance at P the freely propagating assumption is needed. An incremental change in the strength of the C_- disturbance $\delta(\delta u_-)$ as it moves along the C_- characteristic from Q to P is given by (2.8), with $\delta_1 u_+ = 0$ for the freely propagating description. Further, in the absence of C_+ disturbances, the density variation along the C_- characteristic from Q to P is due solely to C_0 disturbances, and is measured in the opposite sense, $\delta\rho = -\delta\rho_C$. Hence (2.8) becomes

$$\delta(\delta u_-) = -\frac{1}{4\rho}\delta\rho\,\delta u_-, \tag{2.12}$$

and after integration from Q to P gives

$$\delta u_-(P) = \left\{\frac{\rho(Q)}{\rho(P)}\right\}^{\frac{1}{4}}\delta u_-(Q). \tag{2.13}$$

The results for the strengths of the disturbances at P involve the density at P, which is now found by a further integration along the C_- characteristic from P to Q. As already noted, the density variation along this path in the freely propagating description is due only to the C_0 disturbances $\delta\rho_C$. Hence an integration of (2.11) starting at P and R and terminating with P and R both at Q yields

$$\rho(P) = \rho(Q)\left\{\frac{\rho(R)}{\rho(Q)}\right\}^{\alpha}. \tag{2.14}$$

For use in §4 we also note the results for fluid and sound speed in the freely propagating description. Proceeding along the C_- characteristic from Q on the shock, only C_0 disturbances are encountered. Hence there is no variation in the pressure or fluid speed along a C_- characteristic in the freely propagating description, and use of (2.14) for the density variation yields

$$u(P) = u(Q), \quad a(P) = a(Q)\left\{\frac{\rho(Q)}{\rho(R)}\right\}^{\frac{1}{2}\alpha}. \tag{2.15}$$

The strengths of the C_0 and C_- disturbances at P, given by (2.11) and (2.13), may, with the help of (2.14), (2.5) and (2.1), be rewritten finally in terms of the density behind the shock as

$$\left.\begin{aligned} \delta\rho_C(P) &= \alpha\left\{\frac{\rho(Q)}{\rho(R)}\right\}^{1-\alpha}\delta\rho(R),\\ \delta u_-(P) &= -\frac{2}{\gamma+1}\beta U(Q)\left\{\frac{\rho(Q)}{\rho(R)}\right\}^{\frac{1}{4}\alpha}\frac{\delta\rho(Q)}{\rho(Q)}.\end{aligned}\right\} \tag{2.16}$$

These results for the strengths of the C_0, C_- disturbances, together with the description of the shock motion in (2.4), constitute the freely propagating description of the flow. In the following sections these results will be used to obtain a description of the overtaking wave.

3. The overtaking wave

In this section we consider the overtaking wave and formulate its effect on the motion of the shock. The wave propagates along C_+ characteristics and is generated in the non-uniform flow behind the shock by the interaction of C_- and C_0 disturbances. In §2 the simple freely propagating description, which neglects the C_+ disturbances, was used to find the strengths of the C_- and C_0 disturbances. These approximate results are now used to estimate the strengths of the C_+ disturbances and to determine the modification to the shock path that they produce. A higher-order approximation, in which these values of the strengths of the C_+ disturbances are used to revise the strengths of the C_- and C_0 disturbances, is not considered.

The change in strength $\delta(\delta u_+)$ of a C_+ disturbance, due to its interaction with a C_0 and C_- disturbance, is given by (2.8) as

$$\delta(\delta u_+) = \frac{1}{4\rho}\,\delta\rho_C(\delta u_- - \delta u_+), \tag{3.1}$$

with $\delta\rho_C$ and δu_- given by (2.16) in terms of density increments immediately behind the shock. This equation is to be integrated along the C_+ characteristic which meets the shock at S (see figure 2). The strength of the C_+ disturbance is written in a form similar to (2.16) for the other disturbances as

$$\delta u_+(P) = \frac{\partial u_+(P)}{\partial\rho(S)}\,\delta\rho(S). \tag{3.2}$$

For a given density distribution, any two of the shock points Q, R, S determine the characteristic intersection point P, so that only two of the densities $\rho(Q)$, $\rho(R)$, $\rho(S)$ just behind the shock are independent. This enables integration variables $\rho(R)$, $\rho(Q)$ contained in the first term on the right-hand side of (3.1) and the variables $\rho(R)$, $\rho(S)$ contained in the second term to be transformed to $\rho(Q)$, $\rho(S)$ with

$$\left.\begin{aligned} \delta\rho(Q)\,\delta\rho(R) &= -\frac{\partial\rho(R)}{\partial\rho(S)}\,\delta\rho(Q)\,\delta\rho(S),\\ \delta\rho(R)\,\delta\rho(S) &= \frac{\partial\rho(R)}{\partial\rho(Q)}\,\delta\rho(Q)\,\delta\rho(S). \end{aligned}\right\} \tag{3.3}$$

The minus sign arises in the first equation because, for the assumed monotonic density distribution, the partial derivative in the Jacobian is negative. Equation (3.1) can now be rewritten as a differential equation for $\partial u_+(P)/\partial\rho(S)$, with the help of the above equations and (2.14), (2.16), as

$$\frac{\partial}{\partial\rho(Q)}\left\{\frac{\partial u_+(P)}{\partial\rho(S)}\right\} = \frac{\alpha\beta U(Q)}{2(\gamma+1)\,\rho(Q)\,\rho(R)}\left\{\frac{\rho(Q)}{\rho(R)}\right\}^{\frac{1}{4}\alpha}\frac{\partial\rho(R)}{\partial\rho(S)} - \frac{\alpha}{4\rho(R)}\frac{\partial\rho(R)}{\partial\rho(Q)}\frac{\partial u_+(P)}{\partial\rho(S)}. \tag{3.4}$$

In this equation $\rho(R)$ is an assumed function of $\rho(Q)$ and $\rho(S)$, and to integrate along a C_+ characteristic we note that $\{\rho(R)\}^{\frac{1}{4}\alpha}$ is an integrating factor and $\rho(S)$ is constant. Far from the shock the disturbance has zero strength, and at the shock P, Q and R coincide at S. The solution is

$$\frac{\partial u_+(S)}{\partial\rho(S)} = \frac{\alpha\beta}{2(\gamma+1)}\int\frac{U(Q)}{\rho(Q)\,\rho(R)}\left\{\frac{\rho(Q)}{\rho(S)}\right\}^{\frac{1}{4}\alpha}\frac{\partial\rho(R)}{\partial\rho(S)}\,d\rho(Q), \tag{3.5}$$

with $\rho(Q) = \rho(S)$ at the upper limit. This integral will be evaluated in §4 for a particular functional dependence of $\rho(R)$ on $\rho(Q)$ and $\rho(S)$ arising from the choice of a power-law initial density distribution.

We conclude this section by determining the effect of the overtaking disturbance on the motion of the shock. This problem has been previously considered by the authors (Chisnell 1955; Yousaf 1974). At the beginning of §2 the freely propagating shock law (2.4) was derived by requiring that the pressure and fluid velocity increments behind the shock be supported solely by a C_- disturbance. In the presence of both C_-

and C_+ disturbances the velocity and pressure increments behind the shock, given by (2.1), will be related by

$$\left.\begin{aligned} \delta u_+ + \delta u_- &= \frac{2}{\gamma+1}\delta U, \\ \delta p_+ + \delta p_- &= \frac{2}{\gamma+1}\delta(\rho_0 U^2). \end{aligned}\right\} \tag{3.6}$$

Rewriting δp_- in terms of δu_- by (2.3), and similarly treating δp_+, enables δu_+ to be determined as

$$\delta u_+ = \frac{1}{\gamma+1}\left\{\delta U + \frac{1}{\rho a}\delta(\rho_0 U^2)\right\}. \tag{3.7}$$

The shock equations (2.1) enable this result to be rewritten as

$$\beta\frac{\delta\rho_0}{\rho_0} + \frac{\delta U}{U} = \frac{(\gamma+1)s}{2+s}\frac{\delta u_+}{U}, \tag{3.8}$$

which is the required modification of the freely propagating law (2.4). Rewriting δu_+ in terms of a density increment, and making further use of (2.1) and (2.2), leads to

$$\left.\begin{aligned} \beta^*\frac{\delta\rho_0}{\rho_0} + \frac{\delta U}{U} &= 0, \\ \beta^* &= \beta\left(1 - 2\gamma\frac{\rho}{a}\frac{\partial u_+}{\partial\rho}\right), \end{aligned}\right\} \tag{3.9}$$

with the partial derivative given by (3.5). This result shows how the freely propagating shock law $U = \kappa\rho_0^{-\beta}$, has the exponent β modified by an overtaking wave, and is valid for any initial density distribution. It will be applied in §4 to the power-law density distribution.

4. An initial power-law density distribution

The freely propagating shock law of §2 gives a description of the shock motion in terms of the density ahead of the shock and does not require the distribution of the density with distance to be specified. The modified description of the shock motion given in §3, which includes the effect of the overtaking disturbance, does not have this property. This is because the choice of the initial density distribution determines which triplets of C_-, C_0, C_+ characteristics, meeting the shock at Q, R, S respectively, are concurrent. The dependence of the overtaking wave on this geometrical property is shown by the term $\partial\rho(R)/\partial\rho(S)$ occurring in (3.5), the differentiation being at constant $\rho(Q)$. An initial power-law distribution of density is considered

$$\rho_0 \propto (-x)^\lambda \quad (x < 0), \tag{4.1}$$

and the modification to the freely propagating shock law produced by the overtaking wave is computed. For this particular law a comparison with an exact solution is possible.

We begin by studying the geometry of the shock path and the characteristics. The speed of the freely propagating shock is given as $\kappa\rho_0^{-\beta}$ by (2.4), and (4.1) enables U to be expressed in terms of x. A further integration provides the shock path and $U(t)$ as

$$t = A(-x)^{1+\beta\lambda}, \quad U = B(-t)^{-b}, \quad b = \frac{\beta\lambda}{1+\beta\lambda}, \tag{4.2}$$

where A, B are constants, and the shock is assumed to arrive at $x = 0$ at $t = 0$. The fluid and sound speeds immediately behind the shock are related to U by (2.1), giving

$$u = \frac{2}{\gamma+1}B(-t)^{-b}, \quad a = \frac{\gamma-1}{\gamma+1}sB(-t)^{-b}. \tag{4.3}$$

At a general point P in the flow behind the shock the fluid and sound speeds are related to these shock values by (2.15), and after use of (4.1) and (4.2) they may be expressed as

$$u(P) = \frac{2}{\gamma+1}B(-t_Q)^{-b}, \quad a(P) = \frac{\gamma-1}{\gamma+1}sB(-t_Q)^{-b}\left(\frac{t_Q}{t_R}\right)^c, \quad c = \frac{\alpha\lambda}{2(1+\beta\lambda)}, \tag{4.4}$$

where t_Q, t_R are the times at which the shock is at Q and R. The equations of the characteristics are thus

$$\frac{dx(P)}{dt} = \frac{2}{\gamma+1}B(-t_Q)^{-b}\left\{1+\tfrac{1}{2}\epsilon(\gamma-1)\,s\left(\frac{t_Q}{t_R}\right)^c\right\}, \tag{4.5}$$

where $\epsilon = -1, 0, 1$ for the C_-, C_0, C_+ characteristics respectively.

Integration of the C_- and C_0 characteristic equations from Q and R, respectively, determines the co-ordinates of P. An integration of the C_+ characteristics from P to S on the shock then provides a relation between t_Q, t_R and t_S. This result may be expressed in terms of densities behind the shock, enabling $\rho(R)$ to be eliminated from (3.5) and the integration with respect to $\rho(Q)$ at constant $\rho(S)$ to be performed.

No integrals of (4.5) have been obtained, and a simplifying assumption is adopted to avoid extensive computation. The basis for the assumption is that the part of the overtaking C_+ wave which catches up the shock before it reaches the origin is contained within a thin layer behind the shock. For any point P in this layer t_Q/t_R remains reasonably close to unity. In addition, for the three values of γ and the three values of λ considered, c is not large; its largest value being 0·46 for $\gamma = \frac{5}{3}$, $\lambda = 2$. Hence $(t_Q/t_R)^c$ is close to unity, and we assume it to have this value in (4.5). In consequence the slopes of the characteristics at P depend only on t_Q, where Q is the intersection of the C_- characteristic through P with the shock path. This results in the C_- characteristics being straight lines, and leads to a simple t_Q, t_R, t_S relationship which is now derived.

We first relate t_P to t_Q and t_R by considering the C_0 characteristic joining R to P (see figure 2). The C_- characteristic joining a general point V on this C_0 characteristic to W on the shock is

$$x_V - x_W = \frac{2}{\gamma+1}B(-t_W)^{-b}\{1-\tfrac{1}{2}(\gamma-1)\,s\}(t_V-t_W). \tag{4.6}$$

Substituting for x_V in the C_0 characteristic differential equation, and using (4.2) for the shock speed dx_W/dt_W, leads to

$$\frac{dt_W}{dt_V}\left[\tfrac{1}{2}(\gamma+1)-\left\{b\left(\frac{t_V}{t_W}-1\right)+1\right\}\{1-\tfrac{1}{2}(\gamma-1)\,s\}\right] = \tfrac{1}{2}(\gamma-1)\,s. \tag{4.7}$$

Integration of this equation, with V, W going from R to P, Q respectively, gives

$$1-\frac{t_P}{t_Q} = \frac{1}{(m+1)\,s}\left\{\left(\frac{t_R}{t_Q}\right)^{m+1}-1\right\}, \quad m = b\left\{\frac{2}{(\gamma-1)\,s}-1\right\}. \tag{4.8}$$

A similar integration of the C_+ differential equation, again making use of the linear (x,t)-relation provided by the C_- characteristics, provides an alternative form for t_P/t_Q in terms of t_S/t_Q. Equating these two expressions for t_P/t_Q gives

$$\frac{1}{(m+1)s}\left\{\left(\frac{t_R}{t_Q}\right)^{m+1}-1\right\}=\frac{(s-1)}{(m+2)s}\left\{1-\left(\frac{t_S}{t_Q}\right)^{\frac{1}{2}m+1}\right\}, \tag{4.9}$$

and differentiation at constant t_Q yields the required geometrical relation

$$\left(\frac{t_R}{t_Q}\right)^{m+1}\frac{1}{t_R}\frac{\partial t_R}{\partial t_S}=-\tfrac{1}{2}(s-1)\frac{1}{t_S}\left(\frac{t_S}{t_Q}\right)^{\frac{1}{2}m+1}. \tag{4.10}$$

These results for t_R and its derivative enable (3.5) to be recast as a definite integral. The flow variables need to be expressed in terms of the time; the velocities are given by (4.2) and (4.3) and the density immediately behind the shock is seen to be proportional to $t^{b/\beta}$ by use of (2.1), (4.1) and (4.2). Substitution into (3.5) gives

$$\frac{\rho(S)}{a(S)}\frac{\partial u_+(S)}{\partial \rho(S)}=\frac{\alpha b}{2(\gamma-1)s}\int_{-\infty}^{t_S}\left(\frac{t_S}{t_Q}\right)^{1+b-b\alpha/4\beta}\frac{1}{t_R}\frac{\partial t_R}{\partial t_S}dt_Q. \tag{4.11}$$

Elimination of t_R by use of (4.9) and (4.10), followed by a change of integration variable, leads to

$$\frac{\rho(S)}{a(S)}\frac{\partial u_+(S)}{\partial \rho(S)}=CI(\eta),\quad I(\eta)=\int_0^1\frac{y^\eta}{1-\zeta y}dy,\quad y=\left(\frac{t_S}{t_Q}\right)^{\frac{1}{2}m+1},$$

$$C=\frac{\alpha b(s-1)}{2(\gamma-1)s\{s+1+ms\}},\quad \eta=\frac{b(1-\frac{1}{4}\alpha/\beta)}{1+\frac{1}{2}m},\quad \zeta^{-1}=1+\frac{1+(m+1)^{-1}}{(s-1)}. \tag{4.12}$$

This integral provides a measure of the strength of the overtaking wave for an initial power-law density distribution. The effect of the overtaking wave on the motion of the shock is given by (3.9), which shows that the shock speed U becomes $\kappa\rho_0^{-\beta^*}$. The constants C and η in (4.12) may be simplified by using the definitions of α, β, m given in (2.7), (2.4) and (4.8) respectively. There follows

$$\frac{\beta^*}{\beta}=1-DI(\eta),\quad D=\frac{s-1}{(2+s)/b+2(2-\gamma)/(\gamma-1)},\quad \eta=\frac{1-s^{-1}}{2/b+s/\gamma-1}, \tag{4.13}$$

which shows the dependence of β^* on the specific heat ratio γ and density exponent λ through the definitions of s, β, b, ζ given in (2.2), (2.4), (4.2), (4.12) respectively.

Calculations for β^* have been performed for the nine cases $\gamma = 1{\cdot}2$, $1{\cdot}4$, $\frac{5}{3}$ and $\lambda=\frac{1}{2}$, 1, 2, and the results are given in table 1. In each case η is less than $0{\cdot}1$, and its smallness enables the integral to be evaluated in a simple manner. The denominator is expanded as a power series, and after integration term by term the series is differenced with the corresponding series for $\eta = 0$, whose sum is $-\zeta^{-1}\ln(1-\zeta)$. The difference series has a rapid convergence, and at most 7 terms were required to obtain the 5-figure accuracy presented in table 1.

For comparison, the values of the Sakurai similarity exponent β_S and the freely propagating exponent β are also shown. In each case β is greater than the exact β_S, and the modified exponent β^* is less than β_S and much closer to it. The percentage errors in β and β^* as approximations to β_S are also shown in the table, and in all cases the inclusion of the approximate treatment of the overtaking wave improves the accuracy by a factor between 3 and 10. The simple treatment of the overtaking wave

		$\lambda = \frac{1}{2}$	$\lambda = 1$	$\lambda = 2$
$\gamma = 1{\cdot}2$	β_S	0·17498	0·17040	0·16545
	β	0·18301 (+4·6)	0·18301 (+7·4)	0·18301 (+10·6)
	β^*	0·17409 (−0·5)	0·16800 (−1·4)	0·16021 (−3·2)
$\gamma = 1{\cdot}4$	β_S	0·20704	0·20214	0·19667
	β	0·21525 (+4·0)	0·21525 (+6·5)	0·21525 (+9·4)
	β^*	0·20613 (−0·4)	0·19957 (−1·3)	0·19080 (−3·0)
$\gamma = \frac{5}{3}$	β_S	0·22820	0·22335	0·21779
	β	0·23607 (+3·4)	0·23607 (+5·7)	0·23607 (+8·4)
	β^*	0·22732 (−0·4)	0·22081 (−1·1)	0·21175 (−2·8)

TABLE 1. The values of the Sakurai similarity exponent β_S for various values of the ratio of specific heats γ and initial density power-law exponent λ. These values are compared with the corresponding parameter β from the freely propagating shock result (2.4); β is in error by amounts ranging from 3 % to 11 %. The modified exponent β^*, which takes approximate account of the overtaking disturbance, is also given (percentage error in brackets) and reduces the error to between 0·4 % and 3·2 %. In each case β_S lies between β and β^*.

thus provides a significant improvement to the freely propagating description of the shock motion. The value of the freely propagating exponent β and its modified value β^* provide bounds in this particular problem for the exact exponent β_S. It is possible that bounds for the shock path may be available for more difficult physical situations for which no similarity solution exists.

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